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ON PAIRWISE CONNECTEDNESS BITOPOLOGICAL SPACES

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ABSTRACT

In this paper, we shall introduce and study the pairwise connected and pairwise disconnected bitopological spaces and investigates some of their characterizations. Some definitions and examples would be providing in order to establish their properties. Furthermore, the 0-dimensional space and pairwise totally disconnected bitopological spaces are also discussed. Topology is formally a study of topological invariants.

Keywords: Bitopological Space, Pairwise Connected, Pairwise Disconnected, zero-Dimensional, Pairwise Totally Disconnected.

1. INTRODUCTION

In mathematics, a bitopological space is a set endowed with two topologies. Typically, if the set is X and the topologies are τ_1 and τ_2 then the bitopological space is referred as (X, τ_1, τ_2). The notion was introduced by J.C.Kelly in the study of Quasi-metrics. i.e., distance functions that are not required to be symmetric. Quasi-uniform spaces, which are generalizations of quasi-metric spaces, also induce bitopological space. Later work in the area has been done, P.Fletcher, Y.W.Kim, E.P.Lane, C.W.Patty, W.J.Pervin, Ivan L.Reilly and others. The systematic study of bitopological space (a set on which is defined two topologies) was begun by J.C.Kelly who introduced various separation properties into bitopological and obtains generalizations of some important classical results and various other authors have contributed to the development of the theory.

The concept of connectedness in a bitopological spaces was introduced by Pervin. Pervin also introduced by the idea of total disconnectedness in bitopological space and also defined the continuity of maps between such spaces. His was the first study of bitopological spaces as “objects in a category”.

The idea of total disconnectedness is introduced for bitopological spaces and some generalizations of classical results are obtained. The most interesting is perhaps a generalization of the classical result which states that "a compact Hausdorff space is totally disconnected if and only if it has a base whose sets are also closed."

2. BASIC DEFINITIONS

Definition: 2.1

A ***topology*** on a set X is a collection τ of subsets of X having the following properties:

1) ϕ and X are in τ.
2) The union of the elements of any subcollection of τ is in τ.
3) The intersection of the elements of any finite subcollection of τ is in τ.

Definition: 2.2

A set X for which a topology τ has been specified is called a ***topological space***, A topological space is an ordered pair (X, τ_1, τ_2) consisting of a set X and a topology τ on X .

A topological space is a set X together with a collection of subsets of X , called ***open sets***, such that ϕ and X are both open and such that arbitrary unions and finite intersections of open sets are open.

Definition: 2.3

Let A be a subset of a topological space X . Then A is called a ***closed set*** in X if its complement $A^C = X - A$ is open in X.

Definition: 2.4

Let X be a topological space with topology τ. If Y is a subset of X, the collection

$$\tau_Y = \{Y \cap U | U \in \tau\}$$

is a topology on Y , called the ***subspace topology***. With this topology, Y is called a subspace of X ; its open sets consist of all intersections of open sets of X with Y.

Definition: 2.5

Given a subset A of a topological space X,

(i) The ***interior of*** A is defined as the union of all open sets contained in A and it is denoted by $Int\ A$

(ii) The ***closure of*** A is defined as the intersection of all closed sets containing A. and it is denoted by $cl\ A$ or $\overline{A}$.

$$Int\ A \subset A \subset \overline{A}$$

If A is open, $A = Int\ A$; while if A is closed, $A = \overline{A}$.

Definition: 2.6

Let X be a topological space. A ***separation*** of X is a pair U, V of disjoint nonempty open subsets of X whose union is X . The space X is said to be connected if there does not exist a separation of X .

Definition: 2.7

A topological space X is said to be:

(i) ***Disconnected*** if $X = A \cup B$ where A and B are any two non-empty separated sets. Thus, X is ***disconnected*** if $(A \cap \overline{B}) \cup (\overline{A} \cap B) = \phi$ and $X = A \cup B$.

For a example,

Let X be an indiscrete space. Then $\tau = \{\phi, X\}$. So, there is no non-empty, proper subset of X. Hence X is connected.

(ii) ***Connected***if it is not disconnected.

For a example,

Let (X, τ) be an indiscrete space. Then $\tau = \{\phi, X\}$. So, there is no non-empty, proper subset of X. Hence X is connected.

(iii) ***Locally connected*** if it has a basis consisting of open connected subsets of X.

In other words, each point has an open connected neighbourhood.

(iv) ***Totally disconnected*** if given any two distinct points x and y, there exists an open nbd-v of x and there exists an open nbd V of y such that $X = U \cup V$ with $U \cap V = \phi$.

Definition: 2.8

Let A be a subset of a topological space X. An open set U containing A is called an ***open neighbourhood*** of A.

Definition: 2.9

Let (X, τ_1, τ_2) and (Y, u_1, u_2) be two bitopological spaces. A function $f : (X, \tau_1, \tau_2) \to (Y, u_1, u_2)$ is said to be ***pairwise continuous*** if the induced functions $f : (X, \tau_1) \to (Y, u_1)$ and $f : (X, \tau_2) \to (Y, u_2)$ are continuous

Definition: 2.10

A bitopological space (X, τ_1, τ_2) is called ***pairwise*** T_2 ***or pairwise hausdorff space*** if given distinct points x, y of X, there is a τ_1-open set U and a τ_2-open set V such that $x \in U$, $y \in V$, $U \cap V = \phi$.

3. PAIRWISE CONNECTEDNESS

Definition: 3.1

Let (X, τ_1, τ_2) be a bitopological space. Then (X, τ_1, τ_2) is said to be ***pairwise connected*** if X cannot be expressed as the union of two non-empty disjoint sets A and B such that

$$(A \cap \tau_1 clB) \cup (\tau_2 clA \cap B) = \phi \quad \text{................. (1)}$$

If (1), is satisfied, A and B as ***pairwise separated sets.***

If $X = A \cup B$. Where A and B satisfy (1), then X is called a ***pairwise disconnected space.*** It can be written as

$X = A/B$, a separation of X.

Example: 3.1

Show that the pairwise connectedness of (X, τ_1, τ_2) is governed by the connectedness of (X, τ_1) and (X, τ_2).

Proof:

Let $X = \{a, b\}$

τ_1 = the discrete topology on X .

τ_2 = the indiscrete topology on X .

$\Rightarrow (X, \tau_1, \tau_2)$ is pairwise connected

But (X, τ_1) is not connected.

Example: 3.2

Let $X = \{a, b\}$

$\tau_3 = \{X, \phi, \{a\}\}$,

$\tau_4 = \{X, \phi, \{b\}\}$,

$\Rightarrow \quad X = \{a\}/\{b\}$ is a separation of (X, τ_3, τ_4).

$\Rightarrow \quad (X, \tau_3, \tau_4)$ is not pairwise connected

But (X, τ_3) and (X, τ_4) are connected.

Definition: 3.2

A subset Y of a bitopological space (X, τ_1, τ_2) is called ***pairwise connected*** if the space $(Y, \tau_1/Y, \tau_2/Y)$ is pairwise connected.

Theorem: 3.1

If C is a pairwise connected subset of a bitopological space (X, τ_1, τ_2) which has a separation $X = A/B$, then either $C \subset A$ (or) $C \subset B$.

Proof:

Let $X = A/B$.

$X = A \cup B$ and $(A \cap (\tau_1 clB)) \cup (\tau_2 clA \cap B)$............. (1)

(1) $\Rightarrow \quad A \cap B = \phi$

$\Rightarrow \quad A = X - B \quad$ (or) $\quad B = X - A$ .... (2)

$\Rightarrow \quad ((C \cap A) \cap \tau_1 cl(C \cap B)) \cup (\tau_2 cl(C \cap A) \cap (C \cap B))$

$\subset (A \cap (\tau_1 clB)) \cup ((\tau_2 clA) \cap B) = \phi$ (By(1))

$\Rightarrow \quad C = (C \cap A)/(C \cap B)$ is a separation of C .

But C is connected.

$\Rightarrow \quad C \cap A = \phi \quad$ (or) $\quad C \cap B = \phi$

$\Rightarrow \quad C \subset (X - A) \quad$ (or) $\quad C \subset (X - B)$

$\Rightarrow \quad C \subset B \quad$ (or) $\quad C \subset A$

Hence the theorem

Theorem: 3.2

If C is a pairwise connected set and $C \subset E \subset \tau_1 clC \cap \tau_2 clC$ then E is connected.

Proof:

Assume that E is not connected.

$\Rightarrow \quad E = A/B$, is a separation.

$\Rightarrow \quad E = A \cup B$ with $\left(A \cap \tau_1 clB\right) \cup \left(\tau_2 clA \cap B\right) = \phi$ .... (1)

And A and B are non-empty set. .... (2)

By the previous theorem,

$$C \subset A \quad \text{(or)} \quad C \subset B$$

Suppose $C \subset A$.Then $B = B \cap B$

$\subset B \cap E$

$\subset B \cap \tau_2 clA = \phi$ (by (1))

$\Rightarrow \quad \phi \subset B \subset \phi$

$\Rightarrow \quad B = \phi$, a contradiction to (2)

Hence the theorem

Theorem: 3.3

If C is a pairwise connected subset of a bitopological space $\left(X, \tau_1, \tau_2\right)$ if X has a separation $X = A/B$, then $C \subset A$ (or) $C \subset B$.

Proof:

By theorem 3.1, we have

$$\Rightarrow \left(\left(C \cap A\right) \cap \tau_1 - cl\left(C \cap B\right)\right) \cup \left(\tau_2 - cl\left(C \cap A\right) \cap \left(C \cap B\right)\right)$$

$$\subset \left(A \cap \left(\tau_1 - clB\right)\right) \cup \left(\left(\tau_2 - clA\right) \cap B\right) = \phi$$

$$\Rightarrow \quad C = \left(C \cap A\right) \cup \left(C \cap B\right)$$

But C is pairwise connected

Hence $C \cap B = \phi$ (or) $C \cap A = \phi$

Where $C = C \cap A$ (or) $C = C \cap B$

$\Rightarrow \quad C \subset B$ (or) $C \subset A$

Hence the theorem

Theorem: 3.4

Let $\left(X, \tau_1, \tau_2\right)$ be a bitopological space. If every two points of X are contained in some pairwise connected subset of X , then X is pairwise connected.

Proof:

Assume that X is not pairwise connected.

Then $X = A/B$

$\Rightarrow X = A \cup B$,

Where A is a τ_1-open and B is a τ_2-open set with $A \cap B = \phi$

Let $x \in A$ and $y \in B$

By hypothesis,There exists a pairwise connected subset C of X .

Such that $x \in C$ and $y \in C$

By theorem 3.1

$$C \subset B (or) C \subset A$$

$\Rightarrow x, y \in A$ (or) $x, y \in B$

This is contradiction

$\Rightarrow \quad (X, \tau_1, \tau_2)$ is pairwise connected.

Hence the theorem

4. ZERO DIMENSIONAL SPACES

Definition: 4.1

A bitopological space (X, τ_1, τ_2) is said to be ***pairwise0-dimensional*** if opens in (X, τ_1) which are closed in (X, τ_2) form a basis for (X, τ_1) and opens in (X, τ_2).

Example: 4.1

Let X be the real line $\mathbb{R}$.

Let $p(x, y) = 0$ if $x \leq y$,

$p(x, y) = 1$ if $x > y, x, y \in X$.

$q(x, y) = p(y, x)$

τ_1, τ_2 are topologies generated by p and q respectively.

Then $[a, \infty)$ is τ_1 -open and τ_1 -closed, $(-\infty, a)$ is τ_2 -open and τ_1 closed.

Hence (X, τ_1, τ_2) is zero dimensional.

Theorem: 4.1

Every non-empty subspace of a 0-dimensional bitopological space is 0-dimensional.

Proof:

Let (X, τ_1', τ_2') be a non-empty subspace of a 0-dimensional space (X, τ_1, τ_2).

Let $p \in X_1$.

Let U_1 be any τ_1' -open neighbourhood U of p in X .

Then there exists a τ_1' -open neighbourhood U of p in X .

Such that $U_1 = U \cap X_1$.

Since is 0-dimensional,

There exists a set V which is both τ_1 -open and τ_2 -closed in X .

Such that $p \in V \subset U$

Let $V_1 = V \cap X_1$.

Then V_1 is both τ_1' -open and τ_2' closed in X_1 and $p \in V_1 \subset U_1$

Similarly, to shows that for every τ_2' -open neighbourhood W_1 of any point $p \in X_1$,

There exists a set Z_1 which is both τ_2' -open and τ_1' -closed in X_1.

Such that $\quad p \in Z_1 \subset W_1$

Thus (X, τ_1', τ_2') is 0-dimensional.

5. PAIRWISE TOTALLY DISCONNECTED

Definition: 5.1

A bitopological space (X, τ_1, τ_2) is said to be ***pairwise totally disconnected***, if each pair of points of X can be separated by a separation of X. That is, given distinct point x, y of X there is a separation of X.

$X = A/B$, such that $x \in A$ and $y \in B$ or $x \in B$ and $y \in A$.

Theorem: 5.1

If (X, τ_1, τ_2) is pairwise T_0 and pairwise zero dimensional then it is totally disconnected.

Proof:

Let x and y be distinct points in X. Then there is

(i) a τ_1-open set U such that $x \in U, y \notin U$ or

(ii) a τ_2-open set V such that $x \notin V, y \in V$.

Since τ_1 is zero dimensional with respect to τ_2,

In case (i),

There is a τ_1-open, τ_2-closed set G.

Such that $x \in G \subset U$

Then $X = G/(X - G)$, is a separation of X with $x \in G$, $y \in X - G$

So that (X, τ_1, τ_2) is totally disconnected.

Case (ii) follows similarly by interchanging the roles of τ_1 and τ_2.

6. CONCLUSION

A bitopological space (X, τ_1, τ_2) as a richer structure, then topological space. A study of bitopological space is a generalization of the study of general topological space as every bitopological space (X, τ_1, τ_2). Can be regarded as a topological space (X, τ), if $\tau_1 = \tau_2 = \tau$. The notation of a bitopological space used in relation to semi-continuous functions restores sufficient symmetry to enable one to use some of the existing techniques of continuous functions.

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